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STRESS–STRAIN RELATIONSHIP FOR PERIODIC LOADING

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A mechanical model for the inelastic periodic deformation of a material is proposed. According to the model, axial plastic strain results from plastic shears in four planes of maximum shear stresses. The plastic shears in these planes are determined by the deformation strength of the material, which is assumed to be different in different systems of sites and depends on the deformation direction. Deformation strength characteristics of materials are proposed. The model and known deformation strength characteristics allow an analytical description of the strain hardening and softening of materials. An example of calculating strain curves of a hypothetical material under soft loading is given.

Key words: *stress, strain, elasticity, plasticity, hardening, softening, fatigue.*

Analytical stress–strain relations for periodic loading are of special significance for developing fatigue failure criteria of materials. Existing theoretical concepts on the stress–strain relationship used to study fatigue cannot be considered adequate even for uniaxial loading. It is also obvious that purposeful development of a mechanical criterion of fatigue failure is possible only if there is a material model that describes periodic loading in the context of plasticity theory and complex loads. The model is required to develop programs for systematic and comprehensive studies of the fatigue damage accumulation and to represent experimental data.

In the present paper, the process of cyclic loading is considered in the context of the phenomenological theory of plasticity using a mechanical scheme of material deformation [1–3]. According to the model of [1–3], the material is elastically isotropic and statistically homogeneous but consists of anisotropic elements with substantially different deformation strengths.

1. We decompose an arbitrary stress state into an equal triaxial tension and two unlike biaxial stress states:

$$\begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} = \sigma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\sigma_0 - \sigma_y) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\sigma_0 - \sigma_z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Here $\sigma_0 = (\sigma_x + \sigma_y + \sigma_z)/3$ is the hydrostatic stress.

In a similar manner, we write the strain tensor as the sum of tensors:

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\varepsilon_0 - \varepsilon_y) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\varepsilon_0 - \varepsilon_z) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad (1)$$

where $\varepsilon_0 = (\varepsilon_x + \varepsilon_y + \varepsilon_z)/3$.

According to Hooke's law, for an isotropic material, the strain tensor in the principal axes coincident with the principal stress axes is written as

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_0(1-2\nu)}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{(\sigma_0 - \sigma_y)(1+\nu)}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{(\sigma_0 - \sigma_z)(1+\nu)}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

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Under the action of external forces, a solid body accumulates potential elastic strain energy Π. In the absence of losses, the work done to deform an elementary material volume is equal to the potential elastic strain energy Π.

The potential elastic strain energy Π of the elementary volume of an isotropic material can be written as

$$\Pi = \frac{3}{2}\frac{\sigma_0^2}{E}(1-2\nu) + \frac{1+\nu}{E}\Big[(\sigma_0-\sigma_y)^2 + (\sigma_0-\sigma_y)(\sigma_0-\sigma_z) + (\sigma_0-\sigma_z)^2\Big]. \qquad (2)$$

The terms in expression (2) are fractions of the potential energy related to the hydrostatic and unlike loads in two orthogonal directions.

We consider the potential energy (2) for uniaxial stress states that occur in standard fatigue specimens loaded in the x direction, and decompose it into components due to equal triaxial tension and biaxial stress states for an anisotropic material:

$$\Pi = \frac{3}{2}\frac{\sigma_0^2}{E}(1-\nu_{xy}-\nu_{xz}) + \frac{3}{2}\frac{\sigma_0^2}{E}(1+\nu_{xy}) + \frac{3}{2}\frac{\sigma_0^2}{E}(1+\nu_{xz}). \qquad (3)$$

Introducing the corresponding notation, we obtain

$$\Pi = \Pi_0 + \Pi_{xz} + \Pi_{xy}.$$

Here Π_0 is the potential energy related to the hydrostatic component of the loading, Π_{xz} and Π_{xy} are the energies due to distortion of the elementary parallelepiped for the plane unlike stress state – extension in the x direction and compression in the z and y directions, respectively.

Expression (3) implies that the components of the total strain energy depend on Poisson's ratios ν_{xy} and ν_{xz}. The components of the potential energy Π_{xy} and Π_{xz} related to the distortion vanish for Poisson's ratios $\nu_{xy} = -1$ and $\nu_{xz} = -1$, respectively.

Deformation with zero value of the potential energy due to the distortion [Π_{xy} or (and) Π_{xz}] can be interpreted as deformation without elastic shear resistance. In this case, the corresponding Poisson's ratios in expression (3) should be equal to -1.

In the case of inelastic deformation, where the longitudinal plastic strain is the result of slipping over the planes of maximum shear stresses τ_{xy} and τ_{yx}, the equality $\nu_{xy} = -1$ should hold; for slipping over the planes of maximum shear stresses τ_{xz} and τ_{zx}, the equality $\nu_{xz} = -1$ should hold.

Using the well-known representation of the total-strain tensor for inelastic deformation in the form of a sum, we obtain

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x^e & 0 & 0 \\ 0 & \varepsilon_y^e & 0 \\ 0 & 0 & \varepsilon_z^e \end{bmatrix} + \begin{bmatrix} \varepsilon_x^p & 0 & 0 \\ 0 & \varepsilon_y^p & 0 \\ 0 & 0 & \varepsilon_z^p \end{bmatrix}. \qquad (4)$$

In expression (4) and below, the superscripts e and p denote the elastic and plastic components of the total-strain tensor, respectively. Similarly to expression (1), the plastic-strain tensor components are written as

$$\begin{bmatrix} \varepsilon_x^p & 0 & 0 \\ 0 & \varepsilon_y^p & 0 \\ 0 & 0 & \varepsilon_z^p \end{bmatrix} = \varepsilon_0^p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\varepsilon_0^p - \varepsilon_y^p) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\varepsilon_0^p - \varepsilon_z^p) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $\varepsilon_0^p = (\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p)/3$.

For ideal plastic flow, the volume change due to plastic strains of the material vanishes: $\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p = 0$ ($\varepsilon_y^p = -\mu_{xy}\varepsilon_x^p$ and $\varepsilon_z^p = -\mu_{xz}\varepsilon_x^p$), which is equivalent to the equality $\varepsilon_x^p(1-\mu_{xy}-\mu_{xz}) = 0$. This implies that

$$\varepsilon_x^p = 0 \quad \text{or} \quad 1 - \mu_{xy} - \mu_{xz} = 0. \qquad (5)$$

Here μ_{xy} and μ_{xz} are the coefficients of the transverse plastic strains.

Thus, the components of the plastic-strain tensor are related by the equation

$$\begin{bmatrix} \varepsilon_x^p & 0 & 0 \\ 0 & \varepsilon_y^p & 0 \\ 0 & 0 & \varepsilon_z^p \end{bmatrix} = \varepsilon_x^p \left\{ \mu_{xy} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{xz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}.$$

The problem of determining the plastic strains is indeterminate since the last equation in (5) has infinite number of solutions.

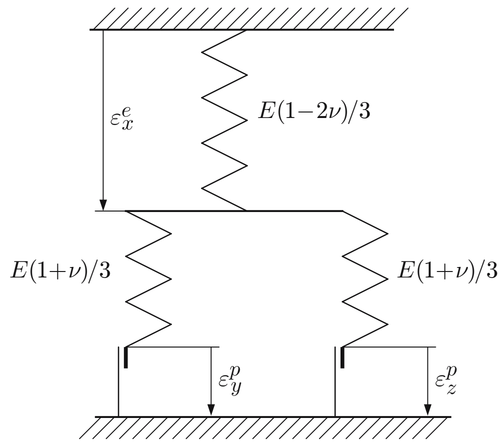

Fig. 1. Rheological model of the material.

2. The model of an elastically isotropic material with different deformation strengths in the planes xy and xz used in the present paper is shown schematically in Fig. 1. The rheological model consists of two components connected in series. The first component — Hooke's element — is connected to the second component by means of two Hooke's elements which work in parallel and are connected to Saint Venant's elements. The force arising in the first elastic element of the model characterizes the hydrostatic stress in the material. The forces arising in the parallel elastic elements are the interaction forces between the octahedral formed in the material by systems of orthogonal sites in which the principal shear stresses act. The relative displacement of the octahedra and the plastic deformation of the material in the loading direction occur in these planes.

Given the characteristics of an elastically isotropic material and its deformation properties, one can construct the deformation model of the material according to the mechanical scheme shown in Fig. 1. Here by the deformation characteristics of the material is meant the conventional elastic and yield limits. These characteristics are determined in tests as stress–strain parameters at the boundaries between the states of elastic strain and partial plasticity and between the states of partial and full plasticity of the material. The state of partial plasticity of the material exists as a transitional state in which elastic and plastic strains in the parallel elements of the model occur simultaneously.

During extension of a specimen of an elastically isotropic material that originally possesses various deformation properties, plastic shears occur not simultaneously at the sites of the principal shear stresses.

Let the shear resistance of the material at the sites xy and yx be lower than that at the sites xz and zx and let the conventional elastic limit σ_x^p and the conventional yield limit σ_x^{fp} be determined from uniaxial tensile tests. The conventional elastic and yield limits exist if the conditions $\varepsilon_y^p = \varepsilon_*$ and $\varepsilon_z^p = \varepsilon_*$ are satisfied, respectively (ε_* is the magnitude of the conventional strain for determining the limit characteristics of the material).

For the deformation of the material whose model is shown in Fig. 1, the following relations hold:

— in the elastic region,

$$\varepsilon_y^p = 0, \qquad \varepsilon_z^p = 0, \qquad \sigma_x \leqslant \sigma_x^p;$$

— in the state of partial plasticity,

$$\Delta\varepsilon_x^e = -\varepsilon_y^p, \qquad \sigma_x^p < \sigma_x < \sigma_x^{fp}$$

($\Delta\varepsilon_x^e$ is the increment in the elastic strain in the specified stress range);

— in the fully plastic state,

$$\Delta\varepsilon_x^{ep} = -\varepsilon_y^{fp}, \qquad \Delta\varepsilon_x^{ep} = -\varepsilon_z^{fp}, \qquad \sigma_x \geqslant \sigma_x^{fp}$$

($\Delta\varepsilon_x^{ep}$ is the increment in the elastic strain and ε_y^{fp} and ε_z^{fp} are the increments in the plastic strain for loading beyond the yield point).

Thus, the total strain can be written as

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x^e & 0 & 0 \\ 0 & -\nu\varepsilon_x^e & 0 \\ 0 & 0 & -\nu\varepsilon_x^e \end{bmatrix} + \begin{bmatrix} -\varepsilon_y^p & 0 & 0 \\ 0 & \varepsilon_y^p & 0 \\ 0 & 0 & \nu\varepsilon_y^p \end{bmatrix} + \begin{bmatrix} -\varepsilon_z^{fp} & 0 & 0 \\ 0 & \varepsilon_z^{fp} & 0 \\ 0 & 0 & \varepsilon_z^{fp} \end{bmatrix} \qquad (6)$$

or

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x^e & 0 & 0 \\ 0 & -\nu\varepsilon_x^e & 0 \\ 0 & 0 & -\nu\varepsilon_x^e \end{bmatrix}$$

$$+ \begin{bmatrix} \Delta\varepsilon_x^e & 0 & 0 \\ 0 & -\Delta\varepsilon_x^e & 0 \\ 0 & 0 & -\nu_{\Delta}\varepsilon_x^e \end{bmatrix} + \begin{bmatrix} \Delta\varepsilon_x^{ep} & 0 & 0 \\ 0 & -\Delta\varepsilon_x^{ep} & 0 \\ 0 & 0 & -\Delta\varepsilon_x^{ep} \end{bmatrix}. \qquad (7)$$

From expressions (6) and (7) it follows that in the material model proposed, the volume change in the elastoplastic region does not vanish and the coefficients of the transverse plastic strains are determined uniquely: $\mu_{xy} = 1$ and $\mu_{xz} = 1$.

3. We study the deformed state of the material subjected to periodic uniaxial loading. For the first loading by a stress σ_x^1, the strains are given by the relations

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^1}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \qquad \text{if} \quad \sigma_x^1 \leqslant \sigma_x^p;$$

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^p}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} + \frac{3(\sigma_x^1 - \sigma_x^p)}{E(2-\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\nu \end{bmatrix} \qquad \text{if} \quad \sigma_x^p < \sigma_x^1 < \sigma_x^{fp}; \qquad (8)$$

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^p}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$+ \frac{3(\sigma_x^{fp} - \sigma_x^p)}{E(2-\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\nu \end{bmatrix} + \frac{3(\sigma_x^1 - \sigma_x^{fp})}{E(1-2\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{if} \quad \sigma_x^1 \geqslant \sigma_x^{fp}.$$

For unloading after the loading beyond the yield limit, formula (8) gives

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^p - \sigma_x^1}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$+ \frac{3(\sigma_x^{fp} - \sigma_x^p)}{E(2-\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\nu \end{bmatrix} + \frac{3(\sigma_x^1 - \sigma_x^{fp})}{E(1-2\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad (9)$$

Expression (9) implies that the total strain of the unloaded material includes elastic and plastic strains of the various elements of the model. It is worth noting that after unloading, the residual strains do not equal to the plastic strains and the elastic strains are partly irreversible.

Loading by a stress of opposite sign $\sigma_x^- = -\sigma_x^1$ beyond the elastic limit ($\sigma_x^{p1} = -\alpha_i^- \sigma_x^p$) and the yield limit ($\sigma_x^{fp1} = -\gamma_i^- \sigma_x^{fp}$) produces the following strains in the compression region:

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^p + \sigma_x^{p1}}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$+ \frac{3(\sigma_x^{fp} - \sigma_x^p + \sigma_x^{fp1} - \sigma_x^{p1})}{E(2-\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\nu \end{bmatrix} + \frac{3(\sigma_x^1 - \sigma_x^{fp} + \sigma_x^- - \sigma_x^{fp1})}{E(1-2\nu)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

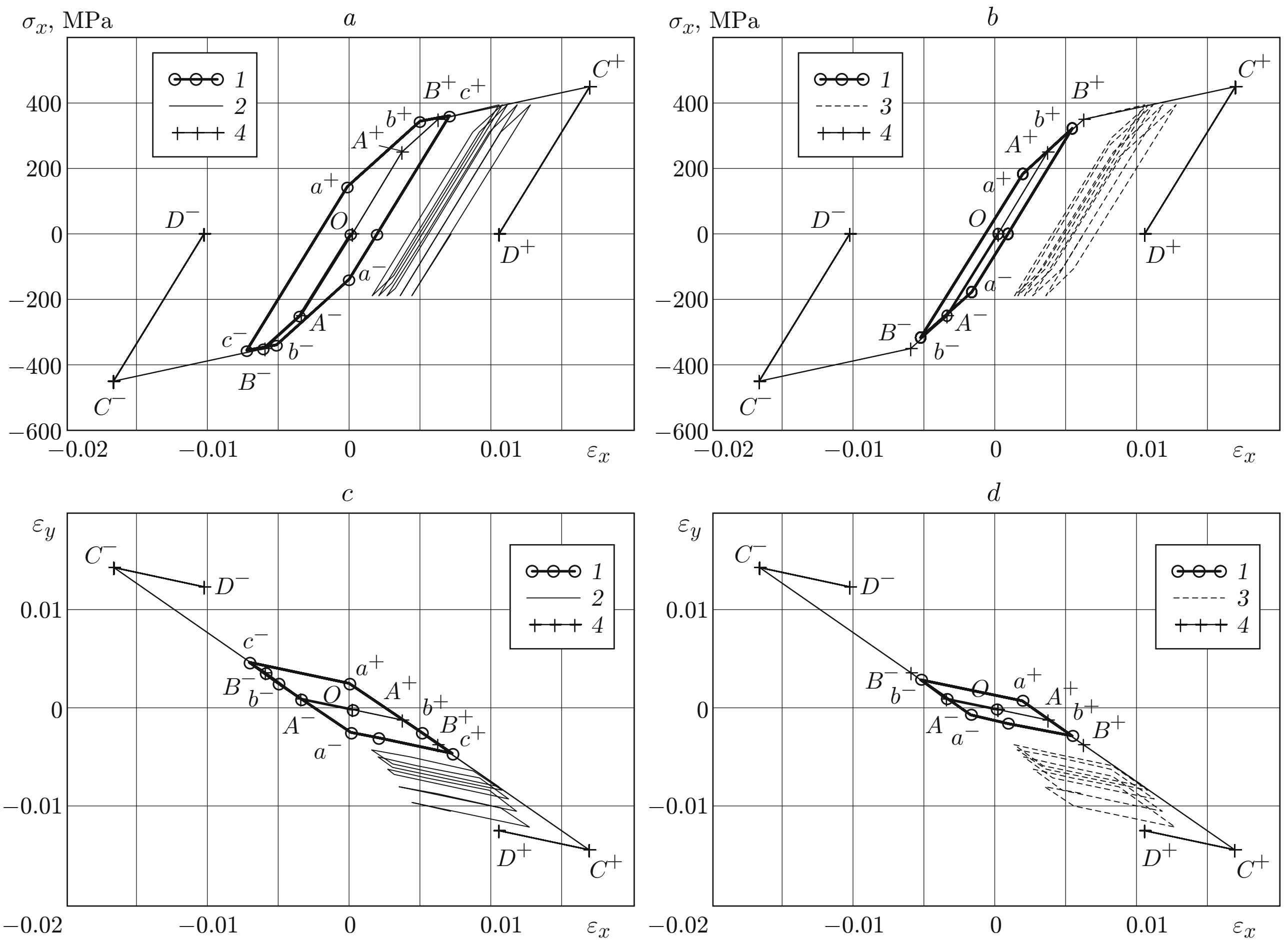

Fig. 2. Curves of cyclic and single deformation $\sigma_x \sim \varepsilon_x$ (a and b) and $\varepsilon_x \sim \varepsilon_y$ (c and d): 1) cyclically ideal material; 2) material with strain-hardening in the region of minimum stresses; 3) material with strain-softening in the region of maximum stresses; 4) single deformation with unloading.

After N deformation cycles with constant parameters of the periodic load, we obtain

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \frac{\sigma_x^p}{E}\Big(1+\sum_{i=1}^{N}(\alpha_i^+ - \alpha_i^-)\Big)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$+\frac{3}{E(2-\nu)}\Big[\sigma_x^{fp}\Big(1+\sum_{i=1}^{N}(\gamma_i^+ - \gamma_i^-)\Big) - \sigma_x^p\Big(1+\sum_{i=1}^{N}(\alpha_i^+ - \alpha_i^-)\Big)\Big]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$+\frac{3}{E(1-2\nu)}\Big[\sigma_x^1 - \sigma_x^{fp}\Big(1+\sum_{i=1}^{N}(\gamma_i^+ - \gamma_i^-)\Big)\Big]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (10)$$

Here α_i^-, γ_i^- and α_i^+, γ_i^+ are coefficients that determine the compressive and tensile elastic and yield limits, respectively (i enumerates the loading cycles starting with the first unloading).

For the constant coefficients $\alpha_i^- = \alpha_i^+ = \gamma_i^- = \gamma_i^+ = 2$, from relations (10), we obtain the solution known in the literature as the Masing principle. For the variable coefficients, we obtain the result known as the generalization of the Masing theory [4] that describes the behavior of cyclically hardened and softened materials.

From relations (10) it follows that strain hardening and softening results from a change in the strain characteristics α_i^+ and γ_i^+ in the region of maximum stresses of the loading cycle and the strain characteristics α_i^- and γ_i^- in the region of minimum stresses.

Figure 2 shows strain curves of a hypothetical material with elastic and strain properties close to those of a D16AT alloy ($E = 7 \cdot 10^4$ MPa, $\nu = 0.3$, $\sigma_x^p = 250$ MPa, and $\sigma_x^{fp} = 350$ MPa) determined by relation (10). The axes in Fig. 2a–d are denoted in the same manner as the components of the stress and strain tensors introduced above.

In Fig. 2, the broken line $OA^+B^+C^+D^+$ is a strain curve for a single deformation by uniaxial tension to point the C^+ with unloading at the point D^+ (no external load).

In Fig. 2a and c, the broken line $OA^-B^-c^-a^+b^+c^+a^-b^-c^-$ is a strain curve for symmetric stress cycles of a cyclically ideal material loaded beyond the yield limit. The broken line originating from the point O represents five deformation cycles with an asymmetric stress cycle for a material with a variable strain characteristic α_i^-, which is responsible for deformation in the region of minimum stresses.

In Fig. 2b and d, the broken line $OA^-b^-a^+b^+a^-$ represents strain curves for a cyclically ideal material loaded by a symmetric stress cycle beyond the elastic limit. The curve originating from point O represents five deformation cycles with an asymmetric stress cycle for a material with a variable strain characteristic α_i^+ responsible for deformation in the region of maximum stresses of the loading cycle.

It follows from Fig. 2 that the calculation results agree qualitatively with the available experimental data.

To calculated strain curves for a real material, it is necessary to obtain additional experimental data on the strain characteristics of the material α_i^+, α_i^-, γ_i^+, and γ_i^- and the stress–strain parameters influencing these characteristics. These characteristics are a subject of further research.

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